Shimura Degrees, New Modular Degrees, and Congruence Primes

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Elliptic Curve Parameterization

- We can parameterize modular elliptic curves by modular curves and Shimura curves.
- It’s often difficult to write down the map, but the degree is accessible.
- We can usually find the optimal quotient.
- This information gives us another way to study all of these objects, and even the related modular forms by way of congruence numbers.
Modular Elliptic Curves

- $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}$
- $X_0(N)$ - the modular curve $\Gamma_0(N) \setminus \mathcal{H} \cup \text{cusps}$
- $J_0(N)$ - Jacobian of $X_0(N)$
- $E$ - a modular elliptic curve over $\mathbb{Q}$ of conductor $N$, with $E = \mathbb{C}/\Lambda$
- $f_E$ - the modular form in $S_2(N)$ associated to $E$ with Fourier coefficients $a_n$. 
Modular Elliptic Curves

$E$ is modular, so we have the following surjective map:

$$
\pi : X_0(N) \to E
$$

given by $\tau \in X_0(N)(\mathbb{C})$

$$
\pi(\tau) = -2\pi i \int_\tau^{i\infty} f(\tau') d\tau' = \sum_{n=1}^{\infty} \frac{a_n}{n} e^{2\pi in\tau} \in \mathbb{C}/\Lambda.
$$
Modular Degree

Let $\pi : X_0(N) \rightarrow E$ be the modular parameterization. We have such a map for any curve isogenous to $E$.

**Definition**
The **modular degree** of $E$ is the minimal such degree.

**Definition**
The **optimal quotient** is the curve $E$ in the isogeny class which gives the minimal degree. Alternatively, the optimal quotient is the curve $E$ in the isogeny class such that the map $J_0(N) \rightarrow E$ has connected kernel.

**Definition**
If $E$ is an optimal quotient of $J_0(N)$, $\pi : J_0(N) \rightarrow E$, $\pi^\vee : E \rightarrow J_0(N)$ $\pi \circ \pi^\vee \in \text{End}(E)$ is multiplication by an integer $m_E$. This integer $m_E$ is called the **modular degree** of $E$. 
Shimura Curves

Let $F$ be a totally real number field. Fix $B$ an indefinite quaternion algebra over $F$ of discriminant $D$ and $\mathcal{O} \subset B$ an Eichler order of level $M$.

- Define $\Gamma_0^D(M)$ to be the group of norm-1 units in $\mathcal{O}$.
- Our Shimura curve is $X_0^D(M) = \Gamma_0^D(M) \backslash \mathcal{H}$.
- We denote its Jacobian by $J_0^D(M)$. 
Quaternionic Modular Forms

Definition
A *quaternionic modular form* of weight $k$ on $\Gamma^D_0(M)$ is a holomorphic function $f$ on $\mathcal{H}$ such that

$$f(\gamma \tau) = (c \gamma + d)^k f(\tau)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^D_0(M)$. The space of such forms is denoted by $M^D_k(M)$, and cusp forms by $S^D_k(M)$. 
New and Old spaces

Let $D$, $M$, and $N$ be positive integers such that $N = DM$ (or ideals in a totally real number field $F$). Then for $f(\tau) \in S_2(M)$ and $r \mid D$, $f(r\tau) \in S_2(N)$. Thus we have maps $S_2(M) \to S_2(N)$ for each $r \mid D$. Combining these maps gives

$$\phi_M : \bigoplus_{r \mid D} S_2(M) \to S_2(N).$$

**Definition**

The image of $\phi_M$ is called the **D-old subspace** $S_2(N)^{D-old}$. The orthogonal complement of $S_2(N)^{D-old}$ in $S_2(N)$ with respect to the Petersson inner product is called the **D-new subspace** $S_2(N)^{D-new}$. 
Jacquet-Langlands correspondance

Theorem (Eichler-Shimura-Jacquet-Langlands)

*There is an injective map of Hecke modules*

\[ S_2^D(M) \hookrightarrow S_2(N) \]

*where* \( N = DM \), *whose image consists of those cusp forms which are new at all primes* \( p \mid D \). *In general there is a non-canonical isomorphism*

\[ S_2^D(M) \cong S_2(N)^{D-\text{new}}. \]

*Working over* \( \mathbb{Q} \), *let* \( J_0^{D-\text{new}}(N) \) *be the* \( D \)-new part of \( J_0(N) \).

**Corollary**

*The Jacobians* \( J_0^{D-\text{new}}(N) \) *and* \( J_0^D(M) \) *are isogenous.*
Degree of Parameterization

We have a parameterization for both $J$-new and Shimura Jacobians.

- $E$ - a modular elliptic curve defined over $F$ of conductor $N$.
- $J$ - either $J_0^D(M)$ (or $J_0^{D\text{-new}}(N)$).
- $\pi : J \to E$ where $E$ is the optimal quotient.
- The **Shimura degree (or D-new degree)** is the degree of $\pi$.

Definition

The endomorphism $\pi \circ \pi^\vee \in \text{End}(E)$ is multiplication by an integer. This integer is called the **Shimura degree (or D-new degree)**, $\delta^D(M)$ (or $\delta^{D\text{-new}}(N)$), of the elliptic curve $E$. 
Idea for studying Shimura Degrees

- Examine character groups of $E$ and $J$ locally, i.e., at primes dividing $N = DM$.

- Use a short exact sequence of Grothendieck to rewrite the degree of parameterization in terms of computable invariants.

- Use dual graphs to view character groups as Hecke modules.

- Use Ribet’s level-lowering sequence to compute Shimura degrees and make comparisons.
Local objects
Let $A$ be a principally polarized abelian variety over $F$ (either $J$, Shimura jacobian or new-modular jacobian, or $E$, elliptic curve) and $p \mid N = DM$:

- $\mathcal{A}_p$ - Néron model
- $\Phi_p(A) = \mathcal{A}_p / \mathcal{A}_p^0$ - Component Group
- $\mathcal{T}_p(A)$ - Toric part of $\mathcal{A}_p$
- $\mathcal{X}_p(A) = \text{Hom}(\mathcal{T}_p(A), \mathbb{G}_m)$ - Character Group

**Theorem (Grothendieck)**

*There is a natural exact sequence*

$$0 \to \mathcal{X}_p(A) \xrightarrow{\alpha} \text{Hom}(\mathcal{X}_p(A), \mathbb{Z}) \to \Phi_p(A) \to 0$$

*in which $\alpha$ is obtained from the monodromy pairing $u_{A,p}$ by $(\alpha(x))(y) = u_{A,p}(x, y)$.***
Alternate Description of Shimura Degree

$A \mapsto \mathcal{X}_p(A)$ is functorial, so induces maps:

$$\pi^* : \mathcal{X}_p(E) \to \mathcal{X}_p(J)$$

$$\pi_* : \mathcal{X}_p(J) \to \mathcal{X}_p(E)$$

then $\pi^* \circ \pi_* : \mathcal{X}_p(E) \to \mathcal{X}_p(E)$ is multiplication by $\delta^D(M)$ on $\mathcal{X}_p(E)$. 
Diagram Chasing

In particular we have

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathcal{X}_p(J) & \rightarrow & \text{Hom}(\mathcal{X}_p(J), \mathbb{Z}) & \rightarrow & \phi_p(J) & \rightarrow & 0 \\
\downarrow & & \uparrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{X}_p(E) & \rightarrow & \text{Hom}(\mathcal{X}_p(E), \mathbb{Z}) & \rightarrow & \phi_p(E) & \rightarrow & 0
\end{array}
\]

As \( \mathcal{X}_p(E) \) injects into \( \mathcal{X}_p(J) \), let \( \mathcal{L}_p(E) \) denote the saturation of \( \pi^* \mathcal{X}_p(E) \). Alternatively,

\[
\mathcal{L}_p(E) = \{ x \in \mathcal{X}_p(J) : T_nx = a_n(f_E)x \text{ for all } n \text{ coprime to } N \}
\]

Note: \( \mathcal{L}_p(E) \) depends only on the isogeny class of \( E \) and not on \( E \) itself.
Formula for Shimura Degree

Let $g_p$ be a generator of $\mathcal{L}_p(E)$ and $\pi_* : \Phi_p(J) \rightarrow \Phi_p(E)$. Define the following notation:

$$h_p = u_{J,p}(g_p, g_p), \quad \bar{c}_p = \#\Phi_p(E), \quad i_p = \#\text{image}(\pi_*), \quad j_p = \#\text{coker}(\pi_*) .$$

Theorem ($F = \mathbb{Q}$ due to Takahashi)

The $\#\text{image}(\pi_*)$ divides $u_J(g_p, g_p)$ and

$$\delta^D(M) = \frac{u_{J,p}(g_p, g_p)}{\#\text{image}(\pi_*)} \cdot \#\text{coker}(\pi_*) = \frac{h_p j_p}{i_p} = \frac{h_p \bar{c}_p}{i_p^2} .$$
Hecke Modules - something we can compute

Let $H$ be the definite quaternion algebra of discriminant $D$ with Eichler order $\mathcal{O}(M)$ of level $M$.

- The **Brandt module** $\text{Br}(D, M) = \mathbb{Z}[\text{Cl}_R(\mathcal{O}(M))]$.
- The **Hecke module** $X(D, M) = \text{Br}(D, M)^0$.
- Computable due to an algorithm of Kirschmir and Voight.
- Inner product:
  \[
  \langle [I], [J] \rangle = \delta_{[I],[J]} \omega_I / 2
  \]
  where $\delta_{[I],[J]} = 1$ if $[I] = [J]$ and 0 otherwise and $\omega_I = \# \mathcal{O}_L(I)^{\times} / \mathbb{Z}_F^{\times}$.
- Hecke operators are matrices with entries:
  \[
  T(p)_{i,j} = \# \left\{ x \in I_i I_j^{-1} : \text{nrm}(xl_i l_j^{-1}) = (p) \right\}.
  \]
Level Lowering Sequence

**Theorem (Buzzard over \( \mathbb{Q} \))**

When \( N = DMp \), \( \mathcal{X}_p(J_0^D(pM)) = \mathcal{X}(Dp, M) \).

**Theorem (Ribet, Buzzard over \( \mathbb{Q} \))**

We have the following short exact sequence of Hecke modules

\[
0 \to \mathcal{X}_p(J_0^{Dpq}(M)) \to \mathcal{X}_q(J_0^D(Mpq)) \to \mathcal{X}_q(J_0^D(Mq)) \times \mathcal{X}_q(J_0^D(Mq)) \to 0.
\]
Computing Character Groups of Jacobians Shimura Curves

There are two cases, $p$ divides the level $p \mid pM$ and $p$ divides the discriminant $p \mid pD$ with $p \mid\mid N = DMp$.

- If $p \mid Mp$: Let $H$ be the definite quaternion algebra of discriminant $pD$ with Eichler order $\mathcal{O}(M)$ of level $M$. Then $\chi_p(J^D_0(M)) \cong X(Dp, M)$.

- If $p \mid Dp$: Let $H$ be the quaternion algebra ramified at all infinite places of discriminant $D$ with Eichler orders $\mathcal{O}(M)$ of level $M$ and $\mathcal{O}(Mp)$ of level $Mp$. Then

$$0 \to \chi_p(J^{Dp,q}_0(M)) \to X(Dq, Mp) \to X(Dq, M) \times X(Dq, M) \to 0.$$
Let $J' = J_{0}^{Dpq}(M)$ and $J = J_{0}^{D}(Mpq)$. Denote invariants of $J'$ with 's.

**Corollary**

$h'_p = h_q$ and $i_q | i'_p$.

**Corollary**

*We have the following relationship between Shimura degrees:*

$$\delta' = \frac{\delta}{C'_p C'_q} i'^2_i 2'$.$p^2.$

**Corollary**

*If we instead let $J' = J_{0}^{D-new}(N)$ and $J = J_{0}(N)$, then $h_p = h'_p$ and $i_p | i'_p$. Further, $m_{E}^{D-new} | m_{E}$.***
How to Compute $h_p$ and $i_p$

The following are now straightforward:

- **$h_p$:** compute the monodromy paring on the generator for $L(f)$ using the action of Hecke operators on $X_p$.

- **$i_p$:** compute the generator of the ideal $I_p$ of $\mathbb{Z}$ by computing the monodromy pairings with $h_p$.

Oddly enough, in most cases this is enough to compute $\delta$ and $\bar{c}_p$'s. Note: If you can compute the optimal quotient, there is an algorithm for finding the Shimura degree.
Data

In fact, for all semistable elliptic curves over $\mathbb{Q}$ with conductor $N < 100$ I can determine both the degree and the optimal quotient:

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Question of Takahashi

**Question (Takahashi)**

If \( p \mid D \), is the map \( \Phi_p(J) \to \Phi_p(E) \) surjective?

If \( p \mid M \), this is not true.

**Corollary (Takahashi)**

Assuming the conjecture, for \( p \mid D \),

\[
\delta^D(M) = \frac{u_J(g_p, g_p)}{\#\text{image}(\pi_\ast)} = \frac{h_p}{i_p}.
\]

**Note:** When working over \( \mathbb{Q} \) this is always enough to compute \( \delta^D(M) \) and find the optimal quotient!
Definition
We say $E$ and $E'$ are discriminant twins if $E$ and $E'$ if $N(E) = N(E')$ and $\Delta(E) = \Delta(E')$, i.e., $E$ and $E'$ have the same conductor and the same discriminant.

Theorem (D. - Lundell)

Over $\mathbb{Q}$ there are only finitely many pairs of semistable, isogenous discriminant twins. They occur for conductors 11, 17, 19, and 37.

Corollary

Assuming Takahashi’s question, over $\mathbb{Q}$ there is an algorithm for finding the Shimura degree and the optimal quotient of $J_0^D(M)$. 
Using power series expansions of quaternion modular forms

- Zagier computed the complex periods of the optimal quotient directly using the Fourier series expansion of the modular form.
- Quaternionic modular forms don’t have cusps, so don’t have Fourier series expansions.
- Voight and Willis use power series expansions instead!
- Compute the power series expansion of the quaternionic modular form \( f_E \in S_2(\Gamma_D^0(M)) \).
- Compute generators for the fundamental domain of \( \Gamma^D_0(M) \).
- Use the generators to identify vertices of the fundamental domain.
- Integrate over vertices to find independent periods.
- Compute the \( j \)-invariant and match with curve in the isogeny class. This curve is the optimal quotient.
Example

Let $F = \mathbb{Q}(\sqrt{5})$, $a = \frac{1+\sqrt{5}}{2}$ and $E : y^2 + xy + ay = x^3 + (-a - 1)x^2$, $N = (-5a + 3)$ of norm 31.

- $X_0^N(1)$ is a genus one curve, so the modular degree is trivially 1.
- There are 6 curves in the isogeny class.
- Using the method of Voight and Willis, compute the $j$-invariant $j(E) = (-a)(-51a + 37)^3(-39a + 25)^3(5a - 3)^{-8}$ and find:

$$E : y^2 + xy + ay = x^3 - (a + 1)x^2 - (30a + 45)x - (111a + 117)$$

- Only one curve in the isogeny class with $\text{ord}_N(\Delta) = 8$, so we find this curve computing Hecke modules as well.
**$\mathbb{Q}(\sqrt{5})$ Example**

Take $N = -8a + 2$, then $\dim M_{(2,2)}(-8a + 2) = 2$ and $N(-8a + 2) = 76$. Let $E$ be the elliptic curve $76a.a1$. Let $X_{0D}(M)$ be the Shimura curve with $D = 2$ and $M = -4a + 1$.

Case $p \mid D$, so $p = 2$. Computing Brandt Modules: $2 = u_J(g_p, g_p)$, $i_p = 1$ so $\delta = 2\bar{c}_2$. Two choices for $\bar{c}_2$, 1 and 5, so $\delta = 2$ or 10.

Try $p \mid M$, so $p = -4a + 1$ Use the Hecke module correspondence to get again get $\bar{c}_{-4a+1} = 1$ or 5 and again $\delta = 2$ or 10.

Problem: For both curves in the isogeny class $\bar{c}_2 = \bar{c}_{-4a+1}$. 
Modular Degree and Congruence Numbers

Let \( S = S_2(\Gamma_0(N), \mathbb{Z}) \) be the space of weight 2, level \( N \), cuspforms with integral Fourier coefficients. Let \( L = (f_E)^\perp \cap S \).

**Definition**

The **congruence number** \( r_E \) is the integer that satisfies the following equivalent conditions:

- \( r \) is the largest integer such that there exists \( g \in L \) with \( f \equiv g \pmod{r} \).
- \( \{(f, h) | h \in S\} = r^{-1}(f, f)\mathbb{Z} \).
- \( r \) is the order of the finite group \( S/(\mathbb{Z}f + L) \).

**Theorem (Ribet)**

\( m_E \mid r_E \).
Modular Degree and Congruence Numbers

Zagier computed $m_E$ for $N = p$. In all of these examples $m_E = r_E$. This lead to Frey and Muller asking if it is always the case that $m_E = r_E$.

Stein, Agashe, investigate and found, no, not even close. Example: The elliptic curve with Cremona label 54b1 has $m_E = 2$ and $r_E = 6$.

**Theorem (Agashe,Ribet,Stein-2009)**

$m_E \mid r_E$ and if $\text{ord}_p(N) \leq 1$ then $\text{ord}_p(r_E) = \text{ord}_p(m_E)$.

For $\Gamma_1(N)$ they find examples where $m_E \nmid r_E$, in particular 54b1 and also for a curve of squarefree conductor, $N = 38$.

They also note that the analogous statement does not hold for modular abelian varieties, but get a different statement in terms of the exponents of the groups.
### $\mathbb{Q}(\sqrt{5})$ Degrees and Congruence Primes

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*D*-new parts

Let $S = S_2(\Gamma_0(N), \mathbb{Z})^{D-\text{new}}$ and $L = (f) \perp \cap S$.

**Definition**

The *D*-new congruence number $r_D^{\text{new}}$ is the integer that satisfies the following equivalent conditions:

- $r$ is the largest integer such that there exists $g \in L$ with $f \equiv g \pmod{r}$.
- $\{(f, h) | h \in S\} = r^{-1}(f, f)\mathbb{Z}$.
- $r$ is the exponent of the finite group $S/(\mathbb{Z}f + L)$. 
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<th>( M )</th>
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<td>1</td>
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<tr>
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<td>26</td>
<td>( a_1 )</td>
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<tr>
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<td>1</td>
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<tr>
<td>26b ( b )</td>
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<td>( b_1 )</td>
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<td>–</td>
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<tr>
<td>26b ( b )</td>
<td>26</td>
<td>1</td>
<td>( b_2 )</td>
<td>2</td>
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<tr>
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<td>( a_3 )</td>
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</table>
Conjectures

Computing $m_{E}^{D}\text{-new}$ is just a few lines using modular symbols and is very fast compared to computing Brandt modules.

Conjecture

For semistable elliptic curves the following invariants are equal:

$$\delta^{D}(M) = m_{E}^{D\text{-new}} = r_{E}^{D\text{-new}}.$$  

If this is true, it gives more evidence of Takahashi’s conjecture:

Conjecture

When $p \mid D$, $\phi_{p}(J) \to \phi_{p}(E)$ is surjective.

And, as $m_{E}^{D\text{-new}} \mid m_{E}$:

Conjecture

$$\delta^{D}(M) \mid m_{E}.$$
Open Questions

- We can use the work of Voight and Willis to find the $j$-invariant of the optimal quotient of the Shimura curve parameterization up to some precision. Is there an algebraic way to find the optimal quotient? This would give a provable algorithm for computing the Shimura degree.

- For totally real number fields, do we get the same analogues? Does $\delta^D(M) \mid r_E$? When $p \mid D$ is the map on component groups surjective?

- Are there only finitely many semistable, isogenous discriminant twins over totally real number fields? Data indicates yes, but the proof over $\mathbb{Q}$ does not generalize.
Thank you!